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General theory of defects in continuous media

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Abstract

A new general theory of defects in continuous media is introduced. The general mechanisms of generation and healing of defects are established. The kinematic description of continuum media with defects is presented. The definition of defects of different levels is given, and the classification of continuous media with defects is introduced. The hierarchic structure of the theory of defects is established and discussed. It is shown that all the known types of defects are naturally included in the presented classification of defects. A new broad class of defects of new types is established and interpreted. It is shown that the existence of new classes of defects is directly connected with some known theoretical and experimental data on the possibility of generation of such defects as dislocations and disclinations. In particular, it is shown that the generation of dislocations is necessarily connected with the existence of disclinations. The formal class of defects being a source of disclinations is specified. A formal generalization of classification of defects is developed to include the defects of arbitrary finite level. The development of consistent theory of defects is very important from both, fundamental and applied viewpoints. The potential applications include, in particular, the modeling of dispersed composite materials, porous media, dynamics of surface effects, crackling, cavitation and turbulence.

Keywords: Theory of defects; Kinematics of continuous media; Classification of defects; Dislocations; Disclinations

1. Introduction

Recent advances in mechanics of continuous media with defects are closely related to the developments in our views on strength and plasticity of solids with local disturbances contributing to their overall behavior. Considerable achievements in creation of new technologies and materials are tied to the success in

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experimental and theoretical studies of the atomic structure, properties and behavior of defects, such as dislocations, disclinations, point defects, inclusions and interfaces. From one side, the language of the theory of defects is sufficiently universal means of interaction between the researchers working in mechanics, physics and material science. It allows describing from the common ground the variety of the different scale physical processes in the deformable media. From the other side, it is also known that in many cases the defects are being formed already at the processing stage for a number of new promising nano- and non-crystallic materials like, for example, amorphous crystallic composites, nano-composites, quasi-crystallic, nano-quasi-crystallic and some other materials (Gutkin, 2000). These inherited defects affect the overall operational properties of these materials. That is the reason why the theory of defects became so important topic in the modern research and gained the considerable development. Many practically important phenomenological models in theory of defects have been significantly revised in the recent studies.

Different types of defects are introduced and analyzed, see, e.g., Nabarro (1967) and Kroner (1962, 1982). The achievements in the continuum theory of defects, see, e.g., Kroner (1982), De Wit (1960, 1973), and Kadic and Edelen (1983), proved to be very important for the further studies of plastic deformation and for modeling of media on account of scale effects of different levels. The development of continuum models of defects beyond the classical elasticity appears to be very important for the description not only limited to the short-range interactions typical for the interaction of defects, but also for the modeling of size-dependent effects in elasticity and plasticity. Recently these models were developed in the framework of the gradient elasticity (Gutkin, 2000; Aifantis, 1994, 1999) and gradient plasticity (Fleck and Hutchinson, 1993, 1997, 2001; Gao et al., 1999). It has been shown that the gradient theories are quite effective in the analysis of the media at the nano- and micro-levels.

The kinematics of defects is a basis in development of phenomenological continuum models in the theory of defects. Firstly, the kinematics of defects (inconsistencies) is the most important element in application of the variational methods for the description of the higher order energetically consistent continuum gradient models, see Fleck and Hutchinson (1993, 1997, 2001), Gao et al. (1999), and Mindlin (1964). Indeed, the kinematics of defects allows to establish a set of arguments for the correct formulation of the variation of energy functional. Secondly, the kinematic analysis allows to establish the relation between the different types of defects and to analyze the reasons and conditions for their generation and disappearance (Kroner, 1962, 1982; De Wit, 1960, 1973; Aifantis, 1994, 1999; Fleck and Hutchinson, 1993, 1997, 2001; Gao et al., 1999).

The possibility of generation (or birth) and disappearance (or healing) of the defects of such two levels as dislocations and disclinations in the continuous media has been established theoretically and experimentally, see, e.g., Kadic and Edelen (1983) and Likhachev et al. (1986). It has been also shown experimentally that the dislocations on disclinations can be borne and disappear. We are not familiar at the present time with experimental studies that would establish the sources of disclinations with the similar clarity. Nevertheless it is possible to state that the fact of generation and disappearance of disclinations has been demonstrated experimentally, and therefore the existence of sources of disclinations has been established.

In the present paper, we introduce a new general kinematic theory of defects in continuous media. And we establish the general mechanisms of existence of defects, their generation (or birth) and disappearance (or healing).

The significance of the present work is, in particular, in discovering the interconnection between the developed kinematic models for the continuous media with defects and their role in the hierarchy of multi-scale modeling.

The outline of the paper is as follows. In Section 2, the Cauchy continuous media with and without defects and the scalar potential are introduced and discussed. In Section 3, the Papkovich–Cosserat media are defined, the vector potential and vector field of defects are analyzed. The Saint-Venant continuous media with and without defects, tensor potential and tensor field of defects are introduced and investigated in Section 4. The Nth level media models and tensor potential of Nth rank are defined and considered in Section 5.

That is followed by the classification of the fields of defects developed in Section 6. Section 7 provides the conclusions for the present study.

2. The Cauchy continuous media model scalar potential

2.1. Defectless Cauchy medium

Let us denote the continuous vector of displacements in the domain V by $R_i(M)$. And let us establish the conditions under which displacement vector R_i can be represented as a gradient of a scalar function $D^0(M)$, i.e.,

$$R_i = \frac{\partial D^0}{\partial x_i}. \tag{1}$$

For an arbitrary point M_0 and a variable point M in the domain V, the scalar potential in Eq. (1) is determined in terms of the displacement vector R_i in the following way:

$$D^0(M) = D^0(M_0) + \int_{M_0}^{M} R_i \, \mathrm{d}y_i. \tag{2}$$

The condition of unique determination of the scalar function D^0 by means of the displacement vector R_i in arbitrary point M of the medium under study is equivalent to the condition of independence of the contour integral in Eq. (2) from the integration path:

$$\frac{\partial R_i}{\partial x_j} \ni_{ijk} = 0, \tag{3}$$

where $\ni_{ijk}$ is the permutation symbol.

Note that the vector of curls ω_k is defined by the formula

$$\omega_k = -\frac{1}{2} \frac{\partial R_i}{\partial x_j} \ni_{ijk}.$$

Therefore, the necessary and sufficient condition of existence of the scalar function $D^0(M)$ in Eq. (2) can be interpreted as a condition of absence of curls ω_k:

$$\omega_k = 0. \tag{4}$$

Eq. (4) defines the ‘‘defectless Cauchy continuous medium’’ as such a medium in which curls are absent and the displacement field has a scalar potential.

It is well known that for a formal mathematical description of a continuous medium in the framework of a variational approach (Fleck and Hutchinson, 2001; Mindlin, 1964) it is sufficient to define a list of continuous arguments. For the defectless Cauchy medium the scalar potential $D^0(M)$ can be chosen as a generalized coordinate. Therefore the defectless Cauchy medium is a model with one degree of freedom.

2.2. Cauchy continuous medium with defects

As it was established above, the displacement field in the defectless Cauchy medium is defined as a gradient of a scalar field $D^0(M)$; the contour integral in Eq. (2) does not depend on the integration path; and vector of curls is zero. In other words, the displacement vector in the defectless Cauchy medium can be defined as a general solution of the homogeneous equation (3). That represents a formal property of the defectless Cauchy medium that is equivalent to the absence of curls.

On the contrary to that, the Cauchy continuous medium with the field of defects is characterized by a presence of non-zero curls:

$$\frac{\partial R_i}{\partial x_j}\ni_{ijk} = -2\omega_k. \tag{5}$$

And in this case, the general solution of Eq. (5) for the displacement field will consist of two components:

$$R_i = \frac{\partial D^0}{\partial x_i} + D_i^1, \tag{6}$$

where

$$\frac{\partial D_i^1}{\partial x_j}\ni_{ijk} = -2\omega_k. \tag{7}$$

The first component in Eq. (6) is a general solution of the homogeneous equation (3), and it is the integrable part of displacements. The second component in Eq. (6) is a particular solution of Eq. (5), or Eq. (7). And therefore it defines a part of displacements that is related to the defectness of medium. Function D_i^1 unlike of function $\frac{\partial D^0}{\partial x_i}$, is non-integrable since the integrability condition for this function is not satisfied.

The following formal definition for the potential of displacement field can be given:

$$D = D^0 + D^1. \tag{8}$$

Function $D(M)$ in Eq. (8) defines the scalar field in the Cauchy medium. It is represented as a sum of $D^0(M)$ and some other scalar function $D^1(M)$ that is determined from the particular solution of Eq. (7) $D_i^1(M)$ as follows:

$$D^1 = \int_{M_0}^{M_x} D_i^1\,\mathrm{d}y_i. \tag{9}$$

Scalar field $D^1(M)$ in Eq. (8) defines a field of discontinuities (or jumps) of potential of displacement field, and it is determined from the non-integrable part of displacements D_i^1. Contour integral in Eq. (9) depends on the integration path. By definition, its value at any point M_0, $J^1(M_0) = \oint D_i^1\,\mathrm{d}y_i$ depends on a trajectory of integration.

The scalar function $D^1(M)$ is not differentiable in the common sense, otherwise it would satisfy the homogeneous equation (3). The generalized derivative of the function $D^1(M)$ can be formally defined as follows:

$$\frac{\partial D^1}{\partial x_i} = D_i^1.$$

Let as define now the Cauchy continuous medium with defects as such that the curls $(-2\omega_k)$ are the sources of defects $D^1(M)$. Under the defects in the Cauchy model we will call the discontinuities of the scalar potential of displacement field. Such defects are defining the continuous field D_i^1 on which the field of defects J^1 is constructed.

The major feature of the above model of medium with defects is that there is no generation of new defects of the above-indicated type. In other words, the source of curls $\omega_k = -\frac{1}{2}\frac{\partial D_i^1}{\partial x_j}\ni_{ijk}$ is absent. The following differential law of conservation takes place:

$$\frac{\partial(-2\omega_k)}{\partial x_k} = 0.$$

This law can be represented in the integral form:

$$\oiint \omega_k n_k\,\mathrm{d}F = 0.$$

Later expression demonstrates the absence of generation of defects in the considered representative volume of the medium.

Let us note that for the Cauchy medium with defects the generalized coordinates are continuous functions $D^0(M)$ and $D_i^1(M)$. They can serve as the arguments of the corresponding functional. Therefore, the Cauchy medium with defects is a model with four degrees of freedom. In this case, the gradients of the generalized coordinates $\frac{\partial D^0}{\partial x_i}$ and $\frac{\partial D_i^1}{\partial x_j}$ are the generalized velocities of the corresponding kinematic state. It can be also observed that in the particular case when $D^0 = 0$, the Cauchy medium model coincides with the classical model of theory of elasticity, in which the generalized coordinates in the variational description are the components of the displacement vector $R_i \equiv D_i^1$, since $D^0(M) \equiv 0$.

3. The Papkovich–Cosserat continuous media model: Vector potential, vector field of defects

3.1. Defectless Papkovich medium

Consider a continuous medium with a non-symmetrical distortion tensor d_{in}^0 defined as a gradient of some continuous vector field r_i^0

$$d_{ij}^0 = \frac{\partial r_i^0}{\partial x_j}. \tag{10}$$

It is well known that a tensor of a second rank d_{in}^0 can be resolved as follows:

$$d_{in}^0 = \gamma_{in}^0 + \tfrac{1}{3}\theta^0\delta_{in} - \omega_k^0 \ni_{ink},$$

where term $\gamma_{in}^0 + \frac{1}{3}\theta^0\delta_{in}$ defines a symmetrical part of the tensor d_{in}^0, and the term $\omega_k^0\ni_{ink}$ defines its anti-symmetric part; γ_{in}^0 is a deviator tensor or deviatoric strain; $\frac{1}{3}\theta^0\delta_{in}$ is a spherical tensor; and θ^0 is an amplitude of the spherical tensor.

Consider now the homogeneous Papkovich equations that represent the existence conditions for the curvilinear integral in definition of the displacement vector

$$(d_{in}^0)_{,m}\ni_{nmj} = 0. \tag{11}$$

Eq. (11) is the existence criterion for the vector potential of the distortion tensor $d_{in}^0 = \gamma_{in}^0 + \frac{1}{3}\theta^0\delta_{in} - \omega_k^0\ni_{ink}$. This vector potential r_i^0 is the displacement vector. There is a full analogy here with the case of scalar potential for the displacement vector R_i in the defectless Cauchy continuous medium.

We will define a defectless Papkovich medium as a medium with a continuous vector potential of the distortion tensor of deformation. In the defectless Papkovich medium the displacement vector is continuous and the distortion tensor d_{in}^0 is a general solution of the homogeneous equation (11). That corresponds to the case of absence of defects—dislocations. In the general case, the defectless Papkovich medium is the Cauchy medium with a continuous displacement vector. Similarly to the Cauchy medium, the scalar defects are present here since the continuous displacement vector contains both the integrable part $\frac{\partial D^0}{\partial x_i}$, as well as a continuous but non-integrable in the sense of Eq. (2) part (D_1^1):

$$r_i^0 = \frac{\partial D^0}{\partial x_i} + D_i^1.$$

In particular, in the case $D^0 \equiv 0$, the defectless Papkovich medium model coincides with the model of classical theory of elasticity. In this case the displacement vector is continuous, but generally non-integrable in the sense of Eq. (2), i.e., the scalar potential for the displacement field does not exist. In the more special particular case $D_i^1 = 0$ the defectless Papkovich medium is completely defectless since both, the dislocations (the vector defects) and the scalar defects are absent in this case.

3.2. Papkovich–Cosserat continuous medium with defects

In the defectless homogeneous Papkovich medium the distortion tensor is integrable since it can be determined from Eq. (10) by means of integration over the displacement vector, and the integrability conditions (11) are fulfilled.

On the contrary to that, for the Papkovich–Cosserat medium with defects, the distortion tensor of deformation d_{ij} can be represented in the general case as a sum of two parts: the integrable part (d^0_{ij}), and the non-integrable part (D^2_{ij}),

$$d_{ij} = d^0_{ij} + D^2_{ij}, \tag{12}$$

where $d^0_{ij} = \gamma^0_{ij} + \frac{1}{3}\theta^0\delta_{ij} - \omega^0_k\ni_{ijk}$. Note that Eq. (12) is analogous to Eq. (6) written for the Cauchy medium displacement vector.

Following the common procedure let us consider now the non-homogeneous Papkovich equations

$$(d_{in})_{,m}\ni_{nmj} = \left(\gamma_{in} + \tfrac{1}{3}\theta\delta_{in} - \omega_k\ni_{ink}\right)_{,m}\ni_{nmj} = \Xi_{ij}. \tag{13}$$

Here $(d_{in})_{,m} = \left(\gamma_{in} + \frac{1}{3}\theta\delta_{in} - \omega_k\ni_{ink}\right)_{,m}$ is the general tensor filed of curvatures of the model under consideration.

On account of relations (11) and (12), Eq. (13) yields

$$(D^2_{in})_{,m}\ni_{nmj} = \Xi_{ij}. \tag{14}$$

The continuous tensor of "inconsistencies" Ξ_{ij} defines the non-homogeneity of the Papkovich relations. The following differential conservation law is valid for this tensor:

$$\frac{\partial \Xi_{ij}}{\partial x_j} = 0.$$

In order to prove that, we will first apply the divergence operator to the left and right sides of Eq. (13)

$$\frac{\partial}{\partial x_j}\left(\frac{\partial d_{in}}{\partial x_m}\ni_{nmj}\right) = \frac{\partial \Xi_{ij}}{\partial x_j}.$$

In the left-hand side of this expression we have

$$\frac{\partial}{\partial x_j}\left(\frac{\partial d_{in}}{\partial x_m}\ni_{nmj}\right) = \frac{\partial^2 d_{in}}{\partial x_j\,\partial x_m}\ni_{nmj}.$$

Evidently, the term $\frac{\partial^2 d_{in}}{\partial x_j\,\partial x_m}$ is symmetrical tensor with respect to indexes j and m. From the other side, by definition the permutation symbol $\ni_{nmj}$ is anti-symmetric tensor for the same pair of indexes. Therefore the convolution of tensors $\frac{\partial^2 d_{in}}{\partial x_j\,\partial x_m}$ and $\ni_{nmj}$ in indexes j and m is equal to zero, which proves the above conservation law.

The solution of the above non-homogeneous Papkovich equation (13) with respect to γ_{ij}, ω_k and θ can be represented as a sum of the following general solution of the homogeneous Papkovich equation for γ^0_{ij}, ω^0_k, θ^0:

$$\gamma^0_{ij} = \frac{1}{2}\frac{\partial r^0_i}{\partial x_j} + \frac{1}{2}\frac{\partial r^0_j}{\partial x_i} - \frac{1}{3}\frac{\partial r^0_k}{\partial x_k}\delta_{ij}, \qquad \omega^0_k = -\frac{1}{2}\frac{\partial r^0_i}{\partial x_j}\ni_{ijk}, \qquad \theta^0 = \frac{\partial r^0_k}{\partial x_k},$$

and the partial solution of the non-homogeneous Papkovich equation (13) denoted by γ^{Ξ}_{ij}, ω^{Ξ}_k and θ^{Ξ}. As a result, we can write

$$\gamma_{ij} = \gamma_{ij}^0 + \gamma_{ij}^{\Xi} = \left(\frac{1}{2}\frac{\partial r_i^0}{\partial x_j} + \frac{1}{2}\frac{\partial r_j^0}{\partial x_i} - \frac{1}{2}\frac{\partial r_k^0}{\partial x_k}\delta_{ij}\right) + \gamma_{ij}^{\Xi},$$

$$\omega_k = \omega_k^0 + \omega_k^{\Xi} = \left(-\frac{1}{2}\frac{\partial r_i^0}{\partial x_j}\ni_{ijk}\right) + \omega_k^{\Xi},$$

$$\theta = \theta^0 + \theta^{\Xi} = \left(\frac{\partial r_k^0}{\partial x_k}\right) + \theta^{\Xi}$$

and

$$d_{ij} = d_{ij}^0 + d_{ij}^{\Xi}, \quad d_{ij}^{\Xi} = \gamma_{in}^{\Xi} + \tfrac{1}{3}\theta^{\Xi}\delta_{in} - \omega_k^{\Xi}\ni_{ink}.$$

The partial solutions of non-homogeneous Papkovich equation with respect to distortion tensor d_{ij}^{Ξ}, or with respect to γ_{ij}^{Ξ}, ω_k^{Ξ} and θ^{Ξ}, which is the same, can be considered as the degrees of freedom that are independent of displacements. For the full analogy with the earlier considered case, the distortion tensor $d_{ij}^{\Xi} \equiv D_{ij}^2 = \gamma_{in}^{\Xi} + \frac{1}{3}\theta^{\Xi}\delta_{in} - \omega_k^{\Xi}\ni_{ink}$ can be considered as "generalized displacements" ("plastic distortion"; see De Wit, 1973; Kadic and Edelen, 1983). Since the "inconsistencies" tensor Ξ_{ij} is related to the "generalized displacements" through the following relations:

$$\Xi_{ij} = \left(\gamma_{in}^{\Xi} + \tfrac{1}{3}\theta^{\Xi}\delta_{in} - \omega_k^{\Xi}\ni_{ink}\right)_{,m}\ni_{nmj}, \tag{15}$$

it can be interpreted as the tensor of "generalized strains" for these "generalized displacements".

Using the Cosserat terminology, we can call $\omega_k^0 = -\frac{1}{2}\frac{\partial r_i^0}{\partial x_j}\ni_{ijk}$ as the restricted curl, and ω_k^{Ξ} as a free curl or spin ("plastic curl"; see De Wit, 1973). Analogously we will call γ_{ij}^0 and θ^0 as restricted strains and γ_{ij}^{Ξ}, θ^{Ξ} as the free strains.

Eqs. (11)–(15) describe the kinematics of continuous media with defects of a dislocation type. These relations lead to the following conclusions:

1. The fields of free strains and spins cannot be uniform because in that case $\Xi_{ij} = 0$.
2. The fields of spins have the sources.

Indeed, we can show by making use of Eq. (15) that the field of spins is not a vorticity field. By applying convolution of left and right sides of Eq. (15) with δ_{ij} we obtain

$$\frac{\partial \omega_k^{\Xi}}{\partial x_k} = -\frac{1}{2}\Xi_{kk} \neq 0.$$

At the same time, the following equality takes place for the vortex fields with the vector of curls ω_k^0:

$$\frac{\partial \omega_k^0}{\partial x_k} = 0.$$

Let us call the models of continuous media with vector potential as the Papkovich–Cosserat media. The kinematics of such media has the following structure:

- The displacement field R_i represents a superposition of the following two fields: the continuous field r_i^0 and the field of displacement jumps or discontinuities D_i^2, i.e.,

$$D_i = r_i^0 + D_i^2 = \left(\frac{\partial D^0}{\partial x_i} + D_i^1\right) + D_i^2, \quad D_i^2 = \int_{M_0}^{M} D_{ij}^2\,\mathrm{d}y_j \ \left(D_{ij}^2 = \gamma_{ij}^{\Xi} + k\frac{1}{3}\theta^{\Xi}\delta_{ij} - \omega_k^{\Xi}\ni_{ijk}\right).$$

The classical displacements R_i are determined by the continuous part only, and they can be represented as follows: $R_i \equiv r_i^0$, $r_i^0 = \frac{\partial D^0}{\partial x_i} + D_i^1$.
Unlike of the continuous displacement field R_i, the vector field D_i defines the complete displacement field with account of dislocations (jumps). The defective displacements D_i are the sum of the classical displacements R_i ($R_i \equiv r_i^0$) and the dislocations D_i^2.

- The field of displacement jumps D_i^2 can be expressed in terms of fields of free strains and spins by means of the following relation (analogous to the Chesaro formula):

$$D_i^2 = \int_{M_0}^{M} D_{ij}^2 \,\mathrm{d}y_j = \int_{M_0}^{M_x} \left(\gamma_{ij}^{\Xi} + \frac{1}{3}\theta^{\Xi}\delta_{ij} - \omega_k^{\Xi}\ni_{ijk}\right)\mathrm{d}y_j.$$

- Tensor of "inconsistencies" of displacements Ξ_{ij} is the tensor of dislocations (see De Wit, 1973; Kadic and Edelen, 1983).
- The following differential conservation law takes place for the dislocation tensor:

$$\frac{\partial \Xi_{ij}}{\partial x_j} = 0.$$

- This conservation law can be represented in the integral form as follows:

$$\int\int\int \frac{\partial \Xi_{ij}}{\partial x_j}\,\mathrm{d}V = \oiint \Xi_{ij} n_j \,\mathrm{d}F = 0.$$

- The flux of tensor Ξ_{ij} through the plane of planar contour can be chosen as a measure of defects (dislocations) (see Kroner, 1982; De Wit, 1960, 1973):

$$\int\int_{+} \Xi_{ij} n_j \,\mathrm{d}F = n_j \int\int_{0} \Xi_{ij}\,\mathrm{d}F,$$

where F is a closed surface stretched over the planar contour.

In other words, the flux of tensor Ξ_{ij} through the arbitrary surface stretched over the chosen planar contour is invariant. Therefore, it can be chosen as a measure of dislocations.

It is important to note that one of major features of the Papkovich–Cosserat continuous media is that it is not possible to describe the birth or disappearance of dislocations in the framework of these media models because $\oiint \Xi_{ij} n_j \,\mathrm{d}F = 0$. Therefore, the defects associated with the conserved dislocation tensor Ξ_{ij} cannot be born or disappear (see Kroner, 1982; De Wit, 1973).

There are two levels of defects in the Papkovich–Cosserat media. The defects of the first rank are related to the conserved dislocations and they are defined by the formula

$$D_i^2 = \int_{M_0}^{M} D_{ij}^2 \,\mathrm{d}y_j.$$

The zero-rank defects are associated with the two types of scalar defects. First type is related to the conserved scalar defects defined by a scalar field (9), $D^1 = \int_{M_0}^{M} D_i^1 \,\mathrm{d}y_i$, see the Cauchy media. The second type is related to such scalar defects that have the conserved dislocations as their sources. These later scalar defects are described by the field $D^2 = \int_{M_0}^{M} D_i^2 \,\mathrm{d}y_i$, where $D_i^2 = \int_{M_0}^{M} D_{ij}^2 \,\mathrm{d}y_j$. They, evidently can be born or can disappear because as it was shown earlier, the fields of spins have the sources,

$$\frac{\partial \omega_k}{\partial x_k} = \frac{\partial \omega_k^0}{\partial x_k} + \frac{\partial \omega_k^{\Xi}}{\partial x_k} = \frac{\partial \omega_k^{\Xi}}{\partial x_k} \neq 0.$$

Note that under the generalized defects we mean the discontinuous part of the considered kinematic characteristics of the medium. The corresponding field is called the defectness field.

In the general case of the Papkovich–Cosserat media the defectness fields of different ranks (scalar, vector and tensor) are defined by the following relations:

- Defectness field of a zero-rank (scalar):

$$D = (D^0) + D^1 + D^2, \quad D^1 = \int_{M_0}^{M_x} D_i^1 \,\mathrm{d}y_i, \;\; D^2 = \int_{M_0}^{M_x} D_i^2 \,\mathrm{d}y_i, \;\; D_i^2 = \int_{M_0}^{M_x} D_{ij}^2 \,\mathrm{d}y_j.$$

- Defectness field of a first-rank (vector):

$$D_i = \left(\frac{\partial D^0}{\partial x_i} + D_i^1\right) + D_i^2, \quad D_i^2 = \int_{M_0}^{M_x} D_{ij}^2 \,\mathrm{d}y_j.$$

- A tensor characteristic of the Papkovich–Cosserat media is the tensor field

$$D_{ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial D^0}{\partial x_i} + D_i^1\right) + D_{ij}^2, \quad D_{ij} \equiv d_{ij} \quad \left(D_{ij}^2 = d_{ij}^{\Xi} = \gamma_{ij}^{\Xi} + \frac{1}{3}\theta^{\Xi}\delta_{ij} - \omega_k^{\Xi}\ni_{ijk}\right),$$

here d_{ij} is a distortion tensor in the Papkovich media.However, the tensor field D_{ij} is not a defectness field of a second rank since it does not contain a discontinuous part. First component in the above expression for D_{ij} is a continuous and integrable part of the tensor field,

$$d_{ij}^0 = \frac{\partial}{\partial x_j}\left(\frac{\partial D^0}{\partial x_i} + D_i^1\right), \quad (d_{ij}^0)_{,m}\ni_{nmj} = 0.$$

The second component D_{ij}^2 in the above expression is a continuous non-integrable part, see Eq. (14). It should be noted for a comparison that among the characteristics of the Cauchy medium, apart of a scalar field of defects $D = D^0 + D^1$ (where D^1 is a discontinuous component), is also a vector field $D_i = \frac{\partial D^0}{\partial x_i} + D_i^1$ that is not a defectness field.

The Papkovich–Cosserat media allow two different types of the sources of defects:

1. The sources of a second rank (sources of dislocations) T_{ij}^2 are defined by a tensor $T_{ij} \equiv T_{ij}^2 = \Xi_{ij}$

$$T_{ij} \equiv T_{ij}^2 = \frac{\partial D_{in}^2}{\partial x_m}\ni_{nmj} = \frac{\partial\left(\gamma_{in}^{\Xi} + \frac{1}{3}\theta^{\Xi}\delta_{in} - \omega_k^{\Xi}\ni_{ink}\right)}{\partial x_m}\ni_{nmj}.$$

2. The sources of a first rank (sources of the scalar defects) T_i are defined by the spins, i.e., by a vector which can be obtained through the convolution of the total tensor of deformations D_{nm} with the tensor $\ni_{nmj}$ in indexes n and m:

$$T_i \equiv T_i^1 = \left(\frac{\partial}{\partial x_m}(R_n) + D_{nm}^2\right)\ni_{nmi} \quad \left(R_i \equiv r_i^0, \;\; r_i^0 = \frac{\partial D^0}{\partial x_i} + D_i^1\right).$$

Since $\frac{\partial}{\partial x_m}\left(\frac{\partial D^0}{\partial x_n}\right)\ni_{nmi} \equiv 0$, we get

$$T_i = (T_i)_1 + (T_i)_2 = \frac{\partial D_n^1}{\partial x_m}\ni_{nmi} + D_{nm}^2\ni_{nmi}, \quad (T_i)_1 = \frac{\partial D_n^1}{\partial x_m}\ni_{nmi} = \frac{\partial R_n}{\partial x_m}\ni_{nmi}, \;\; (T_i)_2 = D_{nm}^2\ni_{nmi} = -2\omega_i^{\Xi}.$$

The sources of defects $(T_i)_1$ are related only to the conserved scalar defects D^1, since the vector of restricted curls $(T_i)_1 = \frac{\partial R_n}{\partial x_m}\ni_{nmi}$ in the general case is not zero, see the Cauchy media. These sources of defects satisfy the conservation condition:

$$\mathrm{div}((T_i)_1) = \frac{\partial (T_i)_1}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial R_n}{\partial x_m}\ni_{nmi}\right) = 0.$$

For the sources of defects $(T_i)_2$ the conservation condition is not satisfied:

$$\frac{\partial (T_i)_2}{\partial x_i} = \frac{\partial D^2_{nm}}{\partial x_i}\ni_{nmi} = T_{ii} \neq 0 \quad \left(T_{ij} \equiv T^2_{ij} = \Xi_{ij},\ \frac{\partial \omega^{\Xi}_k}{\partial x_k} = -\frac{1}{2}\Xi_{kk} \neq 0\right).$$

Therefore, these sources of the scalar defects can be born and can disappear.

4. The Saint-Venant continuous media model: Tensor potential, tensor field of defects

In order to construct the models that will allow the birth and disappearance of dislocations, it is necessary to develop the kinematic continuous media models of a higher order. We will call them the Saint-Venant continuous media.

4.1. Defectless Saint-Venant medium

Let us introduce the curvatures: $\gamma_{ijn} = \frac{\partial \gamma_{ij}}{\partial x_k}$, $\theta_j = \frac{\partial \theta}{\partial x_j}$, $\omega_{ij} = \frac{\partial \omega_i}{\partial x_j}$. These tensors formally define a tensor of curvatures of a third order (see Fleck and Hutchinson, 1993, 1997, 2001; Gao et al., 1999; Mindlin, 1964) D_{ijn} $(\omega_{in}, \theta_n, \gamma_{ijn})$, which is a derivative of the distortion tensor, i.e.,

$$(d_{in})_{,j} = D_{ijn}, \qquad d_{ij} = d_{ij}(M_0) + \int_{M_0}^{M} D_{ijn}\,\mathrm{d}x_n. \tag{16}$$

Here d_{in} is the distortion tensor, and $D_{ijn} = \gamma_{ijn} + \frac{1}{3}\theta_n\delta_{ij} - \omega_{qn}\ni_{ijq}$, also $\gamma_{ijn} = \gamma_{jin}$, $\gamma_{kkn} = 0$.

Following the common procedure let us consider the conditions of integrability for the distortion tensor,

$$\frac{\partial D_{ijn}}{\partial x_m}\ni_{nmk} = 0. \tag{17}$$

Eq. (17) represents the existence conditions for the contour integral (or integrability conditions) in the definition of distortion tensor d_{in} in terms of tensor of curvatures D_{ijn}. Let us call relations (17) as the generalized Saint-Venant relations. The integrability conditions represent the existence criterion for the tensor potential of the tensor of curvatures $D_{ijn} = (d_{in})_{,j}$. The role of this tensor potential of a second rank is played by the distortion tensor d_{in}. There is a full analogy here with the case of scalar potential for the vector R_i (the above Cauchy continuous media model) as well as with the case of vector potential for the distortion tensor (the above Papkovich–Cosserat continuous media model).

Let us now prove that Eq. (17) is a generalization of the well-known Saint-Venant's compatibility equations. First we will rewrite Eq. (17) as follows:

$$\frac{\partial\left(\gamma_{ijn} + \frac{1}{3}\theta_n\delta_{ij} - \omega_{qn}\ni_{ijq}\right)}{\partial x_m}\ni_{nmk} = \frac{\partial \gamma_{ijn}}{\partial x_m}\ni_{nmk} + \frac{\partial \frac{1}{3}\theta_n\delta_{ij}}{\partial x_m}\ni_{nmk} - \frac{\partial \omega_{qn}\ni_{ijq}}{\partial x_m}\ni_{nmk} = 0. \tag{18}$$

Allocate in the tensor equation (18) the anti-symmetric in indexes i, j part. Since first two terms are symmetric in these indexes, we get

$$\frac{\partial \omega_{qn}\ni_{ijq}}{\partial x_m}\ni_{nmk} = 0.$$

This equation represents an existence condition for the vector potential ω_i for curvatures ω_{ij} ($\omega_{ij} = \frac{\partial \omega_i}{\partial x_j}$). From the other side, we know that the integrability, and therefore the existence conditions of the vector of spins is given by Saint-Venant's equations. The above equation coincides with the known Saint-Venant's equations if $\frac{\partial \omega_i}{\partial x_j}$ is expressed in terms of the derivatives of the components of tensor of deformations. Consequently, Eq. (17) contains the Saint-Venant equations as a particular case.

It is easy to obtain the following generalized Saint-Venant's equation (i.e., the compatibility equation) for curvatures θ_n, from Eq. (18) by means of symmetrization and allocation of the spherical part in indexes i, j:

$$\frac{\partial \theta_n}{\partial x_m} \ni_{nmk} = 0,$$

as well as the generalized Saint-Venant's equation for the curvatures γ_{ijn},

$$\frac{\partial \gamma_{ijn}}{\partial x_m} \ni_{nmk} = 0.$$

The above generalized compatibility equations are new. They probably fell out of attention of researchers in earlier studies because in the framework of the classical theory there was no need to define the deformations γ_{ij} and θ in terms of the curvatures γ_{ijn}, θ_j.

The generalized compatibility equations represent the existence conditions for the potentials γ_{ij} and θ for the corresponding curvatures $\gamma_{ijn} = \frac{\partial \gamma_{ij}}{\partial x_n}$, $\theta_j = \frac{\partial \theta}{\partial x_j}$.

We will call the media under study as the Saint-Venant continuous media precisely because the generalized Saint-Venant's equations (17) lay the basis for the analysis of their kinematics. In the defectless Saint-Venant media the tensor of curvatures D_{ijn} is integrable in the sense of Eq. (16). The distortion tensor d_{ij} can be determined uniquely from D_{ijn}, since the integrability conditions (17) for D_{ijn} are fulfilled. In the defectless Saint-Venant's media the distortion tensor d_{in} is continuous and the tensor of curvatures D_{ijn} is a general solution of the homogeneous equation (17).

Note that in the defectless Saint-Venant media the generalized disclinations are absent. In these media, similarly to the Papkovich–Cosserat media with defects, only the conserved dislocations D_i^2 can be present (the defects of a first rank), as well as two types of scalar defects D^1 and D^2; D^1 being the conserved scalar defects, and D^2 the scalar defects that can be born and disappear on the conserved dislocations D_i^2.

4.2. Saint-Venant continuous medium with defects—generalized disclinations

In the general case when the integrability conditions (17) are not fulfilled, the following non-homogeneous equation takes place:

$$\frac{\partial D_{ijn}}{\partial x_m} \ni_{nmk} = \Omega_{ijk}. \tag{19}$$

Here Ω_{ijk} is the continuous tensor of "inconsistencies" given by the relation

$$\Omega_{ijk} = \Gamma_{ijk} + \tfrac{1}{3}\Theta_k \delta_{ij} - \Omega_{qk} \ni_{ijq}. \tag{20}$$

Tensor Ω_{ijk} is a reason of non-homogeneity of the generalized Saint-Venant conditions (19). Alternating and balancing Eqs. (19) and (20) with respect to the first two subscripts, we obtain

$$\frac{\partial \omega_{in}}{\partial x_m} \ni_{nmj} = \Omega_{ij}, \tag{21}$$

$$\frac{\partial \theta_n}{\partial x_m} \ni_{nmj} = \Theta_j, \tag{22}$$

$$\frac{\partial \gamma_{ijn}}{\partial x_m} \ni_{nmk} = \Gamma_{ijk}. \tag{23}$$

By continuing and generalizing the common algorithm we can assume that the curvature fields are integrable or non-integrable depending on equality or non-equality to zero of the corresponding tensors of "inconsistencies" Ω_{ij}, Θ_j and Γ_{ijk}. Let us assume that the tensors of "inconsistencies" (De Wit, 1973). Ω_{ij}, Θ_j and Γ_{ijk} are not equal to zero. By the virtue of Eq. (20) these tensors satisfy the following differential conservation laws:

$$\frac{\partial \Omega_{ij}}{\partial x_j} = 0, \qquad \frac{\partial \Theta_j}{\partial x_j} = 0, \qquad \frac{\partial \Gamma_{ijk}}{\partial x_k} = 0.$$

The fields of full curls can be divided into two parts: continuous part and a part of jumps or discontinuities (spins),

$$\omega_i = \left(-\frac{1}{2}\frac{\partial r_n^0}{\partial x_m}\ni_{nmi} + \omega_i^{\Xi}\right) + \omega_i^{\Omega}, \qquad \omega_{ij} = \left(-\frac{1}{2}\frac{\partial^2 r_n^0}{\partial x_j\,\partial x_m}\ni_{nmi} + \frac{\partial \omega_i^{\Xi}}{\partial x_j} + \omega_{ij}^{\Omega}\right).$$

Here $\omega_{ij}^{\Omega 0} = -\frac{1}{2}\frac{\partial^2 r_n^0}{\partial x_j\,\partial x_m}\ni_{nmi} + \frac{\partial \omega_i^{\Xi}}{\partial x_j}$ can be interpreted as the general solution of the homogeneous Saint-Venant's equation (21)

$$\frac{\partial \omega_{in}^{\Omega 0}}{\partial x_m}\ni_{nmj} = 0,$$

and field of jumps ω_{ij}^{Ω}, that can be interpreted as a partial solution of the non-homogeneous Saint-Venant's equation (21)

$$\frac{\partial \omega_{in}^{\Omega}}{\partial x_m}\ni_{nmj} = \Omega_{ij}\left(= -\frac{1}{2}\Omega_{nmj}\ni_{nmj}\right).$$

Analogously the strain fields can be also divided into two parts: continuous part and a part of jumps or discontinuities,

$$\theta = \left(\frac{\partial r_k^0}{\partial x_k} + \theta^{\Xi}\right) + \theta^{\Omega}, \qquad \theta_j = \left(\frac{\partial^2 r_k^0}{\partial x_j\,\partial x_k} + \frac{\partial \theta^{\Xi}}{\partial x_j} + \theta_j^{\Omega}\right)$$

and

$$\gamma_{ij} = \left(\frac{1}{2}\frac{\partial r_i^0}{\partial x_j} + \frac{1}{2}\frac{\partial r_j^0}{\partial x_i} - \frac{1}{2}\frac{\partial r_k^0}{\partial x_k}\delta_{ij} + \gamma_{ij}^{\Xi}\right) + \gamma_{ij}^{\Omega},$$

$$\gamma_{ijk} = \left(\frac{1}{2}\frac{\partial^2 r_i^0}{\partial x_k\,\partial x_j} + \frac{1}{2}\frac{\partial^2 r_j^0}{\partial x_k\,\partial x_i} - \frac{1}{2}\frac{\partial^2 r_q^0}{\partial x_k\,\partial x_q}\delta_{ij} + \frac{\partial \gamma_{ij}^{\Xi}}{\partial x_k} + \gamma_{ijk}^{\Omega}\right).$$

Here $\theta_i^{\Omega 0} = \frac{\partial^2 r_k^0}{\partial x_j\,\partial x_k} + \frac{\partial \theta^{\Xi}}{\partial x_j}$ is the general solution of the homogeneous equation (22)

$$\frac{\partial \theta_n^{\Omega 0}}{\partial x_m}\ni_{nmj} = 0,$$

and field of jumps θ_i^{Ω} is a partial solution of the non-homogeneous equation (22)

$$\frac{\partial \theta_n^{\Omega}}{\partial x_m}\ni_{nmj} = \Theta_j(= \Omega_{nmj}\delta_{nm}).$$

Correspondingly,

$$\gamma_{ijk}^{\Omega 0} = \frac{1}{2}\frac{\partial^2 r_i^0}{\partial x_k\,\partial x_j} + \frac{1}{2}\frac{\partial^2 r_j^0}{\partial x_k\,\partial x_i} - \frac{1}{2}\frac{\partial^2 r_q^0}{\partial x_k\,\partial x_q}\delta_{ij} + \frac{\partial \gamma_{ij}^{\Xi}}{\partial x_k}$$

can be interpreted as the general solution of the homogeneous equation (23)

$$\frac{\partial \gamma^{\Omega 0}_{ijn}}{\partial x_m} \ni_{nmk} = 0,$$

and field of jumps γ^{Ω}_{ijk} can be interpreted as a partial solution of the non-homogeneous equation (23)

$$\frac{\partial \gamma^{\Omega}_{ijn}}{\partial x_m} \ni_{nmk} = \Gamma_{ijk}\left(=\frac{1}{2}\Omega_{ijk}+\frac{1}{2}\Omega_{ijk}-\frac{1}{3}\Omega_{nmk}\delta_{ij}\right).$$

As a result, the particular solution of Eq. (19): D^3_{ijk}can be determined:

$$\frac{\partial D^3_{ijn}}{\partial x_m} \ni_{nmk} = \Omega_{ijk}, \quad D^3_{ijk} = \gamma^{\Omega}_{ijk} + \frac{1}{3}\theta^{\Omega}_k \delta_{ij} - \omega^{\Omega}_{qk} \ni_{ijq}.$$

Note that in the Saint-Venant media it is possible to define a tensor field of a third rank:

$$\begin{aligned} D_{ijk} &= \frac{\partial d^0_{ij}}{\partial x_k} + D^3_{ijk} = \frac{\partial^2 r^0_i}{\partial x_k \partial x_j} + \frac{\partial d^{\Xi}_{ij}}{\partial x_k} + D^3_{ijk} = \left[\frac{\partial^2}{\partial x_k \partial x_j}\left(\frac{\partial D^0_i}{\partial x_i} + D^1_i\right) + \frac{\partial D^2_{ij}}{\partial x_k}\right] + D^3_{ijk} \\ &= \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial D^0_i}{\partial x_i} + D^1_i\right) + D^2_{ij}\right] + D^3_{ijk}, \\ D_{ijk} &\equiv (d_{ik})_{,j} \quad \left(D^2_{ij} = d^{\Xi}_{ij} = \gamma^{\Xi}_{ij} + \frac{1}{3}\theta^{\Xi}\delta_{ij} - \omega^{\Xi}_k \ni_{ijk},\ D^3_{ijk} = \gamma^{\Omega}_{ijk} + \frac{1}{3}\theta^{\Omega}_k\delta_{ij} - \omega^{\Omega}_{qk}\ni_{ijq}\right). \end{aligned}$$

Let us call the defectness field such a field that contains not only a continuous part but also a field of defects. Tensor field D_{ijk} is not a defectness field, similarly to the fact that tensor field of a second rank D_{ij} is not a defectness field in the Papkovich–Cosserat media of a lower level.

The defective fields of different ranks (scalar, vector and tensor) in the Saint-Venant media are defined by the following relations:

- Defectness field of a zero rank (scalar):

$$\begin{aligned} &D = (D^0) + D^1 + D^2 + D^3, \\ &D^1 = \int_{M_0}^{M_x} D^1_i \, \mathrm{d}y_i, \quad D^2 = \int_{M_0}^{M_x} D^2_i \, \mathrm{d}y_i, \quad D^2_i = \int_{M_0}^{M_x} D^2_{ij} \, \mathrm{d}y_j, \\ &D^3 = \int_{M_0}^{M_x} D^3_i \, \mathrm{d}y_i, \quad D^3_i = \int_{M_0}^{M_x} D^3_{ij} \, \mathrm{d}y_j, \quad D^3_{ij} = \int_{M_0}^{M_x} D^3_{ijk} \, \mathrm{d}y_k. \end{aligned}$$

- Defectness field of a first rank (vector):

$$D_i = \left(\frac{\partial D^0}{\partial x_i} + D^1_i\right) + D^2_i + D^3_i.$$

Defectness field of a first rank defines the general defectness field of displacements that includes all types of dislocations:

$$R_i = (r^0_i) + R^{\Xi}_i + R^{\Omega}_i, \qquad D^2_i \equiv R^{\Xi}_i = \int_M^M D^2_{ij}\,\mathrm{d}x_j, \qquad D^3_i \equiv R^{\Omega}_i = \int\left(\int D^3_{ijk}\,\mathrm{d}x_k\right)\mathrm{d}x_j.$$

- Defectness field of a second rank

$$D_{ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial D^0}{\partial x_i} + D^1_i\right) + D^2_{ij} + D^3_{ij}.$$

In the scalar defectness field D the defects are defined by D^1, D^2, D^3; in the vector field D_i by D_i^2, D_i^3. In the tensor defectness field D_{ij} the defects are defined by D_{ij}^3. There is a full analogy here with the case of Papkovich–Cosserat media.

The continuous parts in the right-hand sides of the defectness fields of the first and second ranks are shown in the brackets:

$$\begin{cases} \omega_k = \left(-\dfrac{1}{2}\dfrac{\partial r_i^0}{\partial x_j}\ni_{ijk} + \omega_k^{\Xi}\right) + \omega_k^{\Omega}, \\ \theta = \left(\dfrac{\partial r_k^0}{\partial x_k} + \theta^{\Xi}\right) + \theta^{\Omega}, \\ \gamma_{ij} = \left(\dfrac{1}{2}\dfrac{\partial r_i^0}{\partial x_j} + \dfrac{1}{2}\dfrac{\partial r_j^0}{\partial x_i} - \dfrac{1}{3}\dfrac{\partial r_k^0}{\partial x_k}\delta_{ij} + \gamma_{ij}^{\Xi}\right) + \gamma_{ij}^{\Omega}, \end{cases}$$

$$\begin{cases} \omega_{ij} = -\dfrac{1}{2}\dfrac{\partial^2 r_n^0}{\partial x_j \partial x_m}\ni_{nmi} + \dfrac{\partial \omega_i^{\Xi}}{\partial x_j} + \omega_{ij}^{\Omega}, \\ \theta_j = \dfrac{\partial^2 r_k^0}{\partial x_j \partial x_k} + \dfrac{\partial \theta^{\Xi}}{\partial x_j} + \theta_j^{\Omega}, \\ \gamma_{ijk} = \dfrac{1}{2}\dfrac{\partial^2 r_i^0}{\partial x_k \partial x_j} + \dfrac{1}{2}\dfrac{\partial^2 r_j^0}{\partial x_k \partial x_i} - \dfrac{1}{2}\dfrac{\partial^2 r_q^0}{\partial x_k \partial x_q}\delta_{ij} + \dfrac{\partial \gamma_{ij}^{\Xi}}{\partial x_k} + \gamma_{ijk}^{\Omega}. \end{cases}$$

Let us analyze now the sources of dislocations in the Saint-Venant medium. The sources of dislocations are defined by the anti-symmetric part of the general tensor of curvatures (defectness tensor field) similarly to Eq. (5) for the Cauchy media and Eq. (13) for the Papkovich–Cosserat media. Then

$$(d_{in})_{,m}\ni_{nmj} = D_{inm}\ni_{nmj} = \gamma_{inm}\ni_{nmj} + \tfrac{1}{3}\theta_n\ni_{imj} - \omega_{km}\ni_{ink}\ni_{nmj} = \Xi_{ij}. \tag{24}$$

Note that the meaning of the density of dislocations Ξ_{ij} in Eq. (24) differs in the general case from the meaning of the density of dislocations in the Papkovich–Cosserat medium, Eq. (14). Equality (24) allows to establish a structure of the density of dislocations in the Saint-Venant medium. We have

$$D_{ijk}\ni_{jkq} = \ni_{jkq}\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial D_i^0}{\partial x_i} + D_i^1\right) + D_{ij}^2\right] + D_{ijk}^3\ni_{jkq} = \frac{\partial D_{ij}^2}{\partial x_k}\ni_{jkq} + D_{ijk}^3\ni_{jkq},$$

$$D_{ij}^2 = d_{ij}^{\Xi} = \gamma_{ij}^{\Xi} + \frac{1}{3}\theta^{\Xi}\delta_{ij} - \omega_k^{\Xi}\ni_{ijk}, \quad D_{ijk}^3 = \gamma_{ijk}^{\Omega} + \frac{1}{3}\theta_k^{\Omega}\delta_{ij} - \omega_{qk}^{\Omega}\ni_{ijq}.$$

Therefore, the quantity Ξ_{ij} from (24) can be written as

$$\Xi_{im} = \frac{\partial}{\partial x_n}\left(\gamma_{ij}^{\Xi} + \frac{1}{3}\theta^{\Xi}\delta_{ij} - \omega_k^{\Xi}\ni_{ijk}\right)\ni_{jnm} + \left(\gamma_{ijn}^{\Omega} + \frac{1}{3}\theta_n^{\Omega}\delta_{ij} - \omega_{kn}^{\Omega}\ni_{ijk}\right)\ni_{jnm} = \Xi_{im}^2 + \Xi_{im}^3,$$

where $\Xi_{iq}^2 = \frac{\partial D_{ij}^2}{\partial x_k}\ni_{jkq}$, $\Xi_{iq}^2 = D_{ijk}^3\ni_{jkq}$.

First component in the above equality for the density of dislocations Ξ_{iq} defines the conserved vector field of defects—dislocations, and it coincides with the density of dislocations in the Papkovich–Cosserat medium. The second component Ξ_{iq}^3 is related to the defects of a higher level, i.e., disclinations. This component of the density of dislocations defines the defects that can be born or disappear on the disclinations.

In order to use the unified notation for sources of defects (similarly to the Papkovich–Cosserat media case) we will adopt the following notation:

$$\Omega_{ijk} \equiv T_{ijk} = T^3_{ijk},$$
$$\Xi_{ij} = \Xi^2_{ij} + \Xi^3_{ij} \equiv T_{ij} = (T_{ij})_2 + (T_{ij})_3,$$
$$-2(\omega^0_i + \omega^{\Xi}_i + \omega^{\Omega}_i) = T_i = (T_i)_1 + (T_i)_2 + (T_i)_3.$$

The above relation (24) represents the existence condition for the dislocations. Correspondingly, the right-hand sides of Eq. (24) are the sources of dislocations. Therefore by means of nine non-homogeneous Papkovich equations we can express nine components of the tensor of curvatures ω_{ij} in terms of dislocation tensor Ξ_{ij}, vector θ_m and the remaining curvatures γ_{inm}:

$$\omega_{ji} = \Xi_{ij} - \tfrac{1}{2}\Xi_{kk}\delta_{ij} - \gamma_{inm}\ni_{nmj} - \tfrac{1}{3}\theta_m\ni_{imj}.$$

The existence conditions for jumps (or discontinuities) in full curls ω_i will follow from the generalized Saint-Venant's equations:

$$\frac{\partial\omega_{in}}{\partial x_m}\ni_{nmj} = \Omega_{ij}$$

or

$$\frac{\partial}{\partial x_m}\left[\Xi_{ni} - \frac{1}{2}\Xi_{kk}\delta_{ni} - \gamma_{npq}\ni_{pqi} - \frac{1}{3}\theta_k\ni_{nki}\right]\ni_{nmj} = \Omega_{ij}. \tag{25}$$

The existence conditions for jumps or discontinuities in the free change of volume after excluding the curvature tensor θ_k by means of Eq. (25) and on account of the generalized Saint-Venant equations lead to the generalization of the differential conservation law for dislocations. Indeed, let us express explicitly the curvatures related to the free change of volume from (25)

$$\frac{1}{3}\frac{\partial\theta_j}{\partial x_i} = \left(\Omega_{ij} - \frac{1}{2}\Omega_{kk}\delta_{ij}\right) - \frac{\partial\Xi_{pk}}{\partial x_q}\left(\delta_{ki}\ni_{pqj} - \frac{1}{2}\delta_{ij}\ni_{pqk} - \frac{1}{2}\delta_{pk}\ni_{iqj}\right) + \frac{\partial\gamma_{npq}}{\partial x_m}\left(\ni_{pqi}\ni_{nmj} - \frac{1}{2}\ni_{pqk}\ni_{nmk}\delta_{ij}\right).$$

Substitute the obtained expressions for $\frac{\partial\theta_j}{\partial x_i}$ into the existence conditions for jumps (or discontinuities) in the free change of volume

$$\frac{\partial\theta_n}{\partial x_m}\ni_{nmj} = \Theta_j.$$

We obtain the following chain of equalities, making use of the equality (24):

$$\begin{aligned}\frac{1}{3}\frac{\partial\theta_j}{\partial x_i}\ni_{ji\mu} &= \frac{1}{3}\Theta_\mu = \Omega_{ij}\ni_{ji\mu} - \frac{\partial\Xi_{pk}}{\partial x_q}(\ni_{pqj}\ni_{jk\mu} - \delta_{pk}\delta_{q\mu}) + \frac{\partial\gamma_{npq}}{\partial x_m}\ni_{pqi}\ni_{nmj}\ni_{ji\mu}\\ &= \Omega_{ij}\ni_{ji\mu} - \frac{\partial\Xi_{pk}}{\partial x_q}(\delta_{pk}\delta_{q\mu} - \delta_{p\mu}\delta_{qk} - \delta_{pk}\delta_{q\mu}) + \frac{\partial\gamma_{npq}}{\partial x_m}(\delta_{p\mu}\delta_{qj} - \delta_{pj}\delta_{q\mu})\ni_{nmj}\\ &= \Omega_{ij}\ni_{ji\mu} + \frac{\partial\Xi_{\mu k}}{\partial x_k} + \frac{\partial\gamma_{n\mu j}}{\partial x_m}\ni_{nmj} - \frac{\partial\gamma_{nj\mu}}{\partial x_m}\ni_{nmj} = \Omega_{ij}\ni_{ji\mu} + \frac{\partial\Xi_{\mu k}}{\partial x_k} - \Gamma_{n\mu n}.\end{aligned}$$

By moving the divergence of the dislocation tensor to the left-hand side, we will finally arrive to the following equation:

$$\frac{\partial\Xi_{\mu k}}{\partial x_k} = \Gamma_{n\mu n} + \frac{1}{3}\Theta_\mu - \Omega_{ij}\ni_{ji\mu} = \Omega_{\mu kk}. \tag{26}$$

Eq. (26) transforms into the conservation law for the dislocation tensor in the Papkovich–Cosserat media model in the case of zero right-hand side, i.e., $\frac{1}{3}\Theta_\mu - \Omega_{ij}\ni_{ji\mu} + \Gamma_{n\mu n} = 0$. For non-zero right-hand side this equation will describe the birth and disappearance of dislocations.

Eq. (26) can be represented as follows using the above introduced notation:

$$\frac{\partial T_{jk}}{\partial x_k} = T_{jkk} \quad (\Xi_{ij} \equiv T_{ij},\ \ \Omega_{ijk} \equiv T_{ijk}).$$

We will interpret the field of jumps ω_i^Ω (the Frank vector field) as a vector field of disclinations, and we will call the tensor of "inconsistencies" Ω_{ij} as the tensor of disclinations, following De Wit (1973). Note that unlike of classical view, the dislocations can be borne and disappear even in the absence of disclinations, i.e., $\Omega_{ij} = 0$, if we will take into account existence of two new classes of defects defined by the "inconsistencies" tensors Θ_μ and $\Gamma_{n\mu n}$. These tensors like the disclinations are equally possible sources of dislocations. And the defects defined by these tensors play the same role as the disclinations play in birth and disappearance of dislocations.

Let us call the scalar field of jumps in the change of volume θ^Ω as pores, and the vector of "inconsistency" of free change of volume Θ_j as vectors of cavitation. We will also call the tensor field of jumps in deviator γ_{ij}^Ω as the field of twinning, and the "inconsistencies" tensor Γ_{ijk} as a tensor of twinning.

The Saint-Venant continuous media are described in the general case by 40 degrees of freedom: $D^0 r_i^0$, ω_k^Ξ, θ^Ξ, γ_{ij}^Ξ, ω_{ij}^Ω, θ_j^Ω, and γ_{ijk}^Ω. They allow a three-level system of defects. The zero level of defects includes three types of defects: D^1, D^2, D^3. The first level of defects corresponds to dislocations that may be conserved, as well as they may be borne or disappear. The second level of defects corresponds to the non-conservable disclinations, cavitations and twinnings. The set of the Saint-Venant media contains in itself the sub-sets of the Papkovich–Cosserat media as well as the Cauchy media. The Papkovich–Cosserat media are described by 13 degrees of freedom: $D^0 r_i^0$, ω_k^Ξ, θ^Ξ, and γ_{ij}^Ξ. And the classical Cauchy media are described by only three degrees of freedom r_i^0.

The following models can be constructed in the framework of the Saint-Venant kinematic continuous media model as the particular cases:

- Media with "turbulence" described by 15 degrees of freedom r_i^0, ω_k^Ξ, and ω_{ij}^Ω, in which the spins ω_k^Ξ are conserved but the spins ω_k^Ω may be borne or disappear;
- "Cavitational" media described by seven degrees of freedom r_i^0, θ^Ξ, and θ_j^Ω, in which the pores θ^Ξ are conserved but the pores θ^Ω may be borne or disappear.
- Media with twinning described by 23 degrees of freedom r_i^0, γ_{ij}^Ξ, and γ_{ijk}^Ω, in which the free shifts γ_{ij}^Ξ are conserved but the free distortions γ_{ij}^Ω may be borne or disappear, like it happens for example in the processes of crystallization or polymerization.

5. The *N*th level continuous media model: Tensor potential of *N*th rank

In this section we develop the kinematic model for defects of $(N-1)$th level. For this purpose, we define the conserved tensor of "inconsistencies" $T^N_{\ldots\rho} \equiv T_{\ldots\rho}$ of (N)th rank from the equation

$$\frac{\partial T^N_{\ldots\rho}}{\partial x_\rho} = 0, \tag{27}$$

where ρ is a last Nst subscript of the tensor of "inconsistencies". Then the field of multi-strains $D_{\ldots n}$ of Nth rank will be defined as a general solution of the conservation equation (27)

$$T^N_{\ldots\rho} = \frac{\partial D_{\ldots n}}{\partial x_m} \ni_{nm\rho} \quad (T^N_{\ldots\rho} \equiv T_{\ldots\rho}). \tag{28}$$

Representing the solution of this non-homogeneous equation of compatibility as a sum of the general solution $\frac{\partial D_{\ldots}}{\partial x_n}$ of the homogeneous equation (27) $(\frac{\partial D^N_{\ldots n}}{\partial x_m} \ni_{nm\rho} = 0)$ and a particular solution $D^N_{\ldots n}$ of the non-homogeneous equation (28) we obtain

$$D_{\dots n} = \frac{\partial D_{\dots}}{\partial x_n} + D^{N}_{\dots n}. \tag{29}$$

Here tensor $D_{\dots}$ of $(N-1)$st rank can be interpreted as a tensor potential for some tensor of Nth rank. Let us call this tensor $\frac{\partial D_{\dots}}{\partial x_n}$ as a tensor of restricted (integrable) multi-deformation; and the tensor $D^{N}_{\dots n}$ we will call a tensor of a free (non-integrable) multi-deformation. From the other side, $D_{\dots}$ can be considered as a continuous part of the field of multi-displacements. By representing the field of multi-displacements in the form analogous to the field of multi-strains we can write for the tensor of multi-displacement:

$$D_{..k} = \frac{\partial D^{N-2}_{..}}{\partial x_k} + D^{N-1}_{..k}. \tag{30}$$

Therefore the tensor of multi-displacements is represented as a sum of integrable $\frac{\partial D^{N-2}_{..}}{\partial x_k}$ and non-integrable $D^{N-1}_{..k}$ components. Then, with the account of Eqs. (29) and (30) we can write

$$D_{..k\,n} = \frac{\partial}{\partial x_n}\left(\frac{\partial D^{N-2}_{..}}{\partial x_k} + D^{N-1}_{..k}\right) + D^{N}_{..k\,n}, \qquad D_{\dots n} \equiv D_{..k\,n}, \qquad D^{N}_{\dots n} \equiv D^{N}_{..kn}. \tag{31}$$

Here k is a last but one subscript of the defectness field $D_{..k\,\mathrm{n}}$ of Nth rank or it is a last subscript of the defectness field $D_{..k}$ of $(N-1)$st rank.

The complete field of multi-displacements $D_{..k}$ (defectness filed) can be determined from Eq. (31) by means of the generalized Chesaro formulae in the form of sum of a continuous component of multi-displacements ($\frac{\partial D^{N-2}_{..}}{\partial x_k} + D^{N-1}_{..k}$) and a field of multi-dislocations $D^{N}_{..k}$ (defects of $(N-1)$st rank)

$$D_{..k} = \left(\frac{\partial D^{N-2}_{..}}{\partial x_k} + D^{N-1}_{..k}\right) + D^{N}_{..k}, \qquad D^{N}_{..k} = \int_{M_0}^{M_x} D^{N}_{..k\,n}\,\mathrm{d}y_n.$$

That leads to the formal definition of the "inconsistencies" tensor of a last but one level $T^{N}_{..q}$ of $(N-1)$st rank

$$\frac{\partial D_{..k}}{\partial x_n}\ni_{knq} = \frac{\partial^2 D^{N-2}_{..}}{\partial x_n\,\partial x_k}\ni_{knq} + \frac{\partial D^{N-1}_{..k}}{\partial x_n}\ni_{knq} + D^{N}_{..k\,n}\ni_{knq} = \frac{\partial D^{N-1}_{..k}}{\partial x_n}\ni_{knq} + D^{N}_{..k\,n}\ni_{knq} = T_{..q}$$

or

$$\frac{\partial D_{..k}}{\partial x_n}\ni_{knq} = T_{..q}.$$

The law of birth and disappearance of sources of defects of the last but one level takes the following form:

$$\frac{\partial T_{..q}}{\partial x_q} = \frac{\partial}{\partial x_q}\left(\frac{\partial D^{N-1}_{..k}}{\partial x_n}\ni_{knq} + D^{N}_{..k\,n}\ni_{knq}\right) = \frac{\partial D^{N}_{..k\,n}}{\partial x_q}\ni_{nqk} = T_{..kk}.$$

In result

$$\frac{\partial T_{..q}}{\partial x_q} = T_{..kk}. \tag{32}$$

Here $T_{..kk}$ is a tensor of $(N-1)$nd rank, since it is formed by the convolution in last two indexes of the corresponding tensor of Nth rank, compare with the Saint-Venant media case. Tensor $T_{..kk}$ defines the sources of defects of $(N-1)$nd rank. After N iterations of this algorithm we will arrive to the field of milti-displacements of the zero rank, i.e., to the scalar field D. That will be the natural conclusion of the algorithm.

Note in conclusion that the above-described algorithm can be considered as a realization of the mathematical induction in the construction of the geometrical theory of defects of Nth rank.

6. Classification of the fields of defects

The above introduced algorithm of the kinematic analysis of the fields of defects allows us to introduce the following general classification of kinematic models for continuum media with defects. This classification is in a good agreement with the available experimental data.

1. The facts of generation and healing of defects of up to the second level have been validated, namely:
   - zero level defects with the tensor of "inconsistencies" $T_i = -2\omega_i$ (turbulence as a defect of the potential state of the continuous medium);
   - first level defects with the tensor of "inconsistencies" $T_{ij} = \Xi_{ij}$ (dislocations); and
   - second level defects with the tensor of "inconsistencies" $T_{ijk} = \Omega_{ijk}$ (generalized disclinations—"classical" disclinations, cavitation, twinning).

   Indeed, turbulence is a well studied phenomenon. And the defects of dislocations and disclinations are well established experimentally. In the present paper we have offered the explanation for the processes of generation and healing of defects. And we established the interrelation between these processes.
2. After we established that the processes of generation and healing of disclinations take place, in accordance with the present study we should acknowledge the existence of defects of third level with the conserved tensor of "inconsistencies" $T_{ijnm} = \Theta_{ijnm}$. And therefore it is necessary to take them into account. Otherwise the generalized disclinations could not be born or disappear.
3. The model of continuous media of Nth level with defects has the following kinematic structure:
   - The defectness fields up to the Nth rank inclusive are determined as follows:

$$D = (D^0) + D^1 + D^2 + D^3 + D^4 + \cdots,$$

$$D_i = \left(\frac{\partial D^0}{\partial x_i} + D_i^1\right) + D_i^2 + D_i^3 + D_i^4 + \cdots,$$

$$D_{ij} = \left(\frac{\partial^2 D^0}{\partial x_j \partial x_i} + \frac{\partial D_i^1}{\partial x_j} + D_{ij}^2\right) + D_{ij}^3 + D_{ij}^4 + \cdots,$$

$$D_{ijk} = \left(\frac{\partial^3 D^0}{\partial x_k \partial x_j \partial x_i} + \frac{\partial^2 D_i^1}{\partial x_k \partial x_j} + \frac{\partial D_{ij}^2}{\partial x_k} + D_{ijk}^3\right) + D_{ijk}^4 + \cdots,$$

$$D_{ijks} = \left(\frac{\partial^4 D^0}{\partial x_s \partial x_k \partial x_j \partial x_i} + \frac{\partial^3 D_i^1}{\partial x_s \partial x_k \partial x_j} + \frac{\partial^2 D_{ij}^2}{\partial x_s \partial x_k} + \frac{\partial D_{ijk}^3}{\partial x_s} + D_{ijks}^4\right) + \cdots,$$

where all the expressions in brackets represent the continuous parts of fields.

   - The discontinuous fields of defects are defined by the following equalities:

$$D^1 = \int_{M_0}^{M_x} D_i^1\,\mathrm{d}y_i, \quad D^2 = \int_{M_0}^{M_x} D_i^2\,\mathrm{d}y_i, \quad D^3 = \int_{M_0}^{M_x} D_i^3\,\mathrm{d}y_i, \quad D^4 = \int_{M_0}^{M_x} D_i^4\,\mathrm{d}y_i, \quad \ldots,$$

$$D_i^2 = \int_{M_0}^{M_x} D_{ij}^2\,\mathrm{d}y_j, \quad D_i^3 = \int_{M_0}^{M_x} D_{ij}^3\,\mathrm{d}y_j, \quad D_i^4 = \int_{M_0}^{M_x} D_{ij}^4\,\mathrm{d}y_j, \quad \ldots,$$

$$D_{ij}^3 = \int_{M_0}^{M_x} D_{ijk}^3\,\mathrm{d}y_k, \quad D_{ij}^4 = \int_{M_0}^{M_x} D_{ijk}^4\,\mathrm{d}y_k, \quad \ldots,$$

$$D_{ijk}^4 = \int_{M_0}^{M_x} D_{ijks}^4\,\mathrm{d}y_s, \quad \ldots,$$

$$\ldots$$

In development of the mathematical models of continuous media with defects the above continuous fields $D^0, D_i^1, D_{ij}^2, D_{ijk}^3, D_{ijks}^4, \ldots$, as well as their derivatives can serve as the arguments of the corresponding functionals or the corresponding variational equations.

- The sources of defects of Nth rank satisfy Eqs. (28) and (32). In particular, if we assume that the defects of fourth rank are conserved then the sources of defects (tensor of "inconsistencies") will satisfy the following relations:

$$
\begin{aligned}
&T_{ijkq} = \frac{\partial D_{ijkn}}{\partial x_m} \ni_{nmq} = \frac{\partial D^3_{ijkn}}{\partial x_m} \ni_{nmq}, \quad \frac{\partial T_{ijkq}}{\partial x_q} = 0,\\
&T_{ijk} = \frac{\partial D_{ijn}}{\partial x_m} \ni_{nmk} = \frac{\partial D^2_{ijn}}{\partial x_m} \ni_{nmk} + D^3_{ijnm} \ni_{nmk}, \quad \frac{\partial T_{ijk}}{\partial x_k} = T_{ijnn},\\
&T_{ij} = \frac{\partial D_{in}}{\partial x_m} \ni_{nmj} = \frac{\partial D^1_{in}}{\partial x_m} \ni_{nmj} + D^2_{inm} \ni_{nmj} + D^3_{inm} \ni_{nmj}, \quad \frac{\partial T_{ij}}{\partial x_j} = T_{inn},\\
&T_{i} = \frac{\partial D_{n}}{\partial x_m} \ni_{nmi} = \frac{\partial D^0_{n}}{\partial x_m} \ni_{nmi} + D^1_{nm} \ni_{nmi} + D^2_{nm} \ni_{nmi} + D^3_{nm} \ni_{nmi}, \quad \frac{\partial T_{i}}{\partial x_i} = T_{nn}.
\end{aligned}
$$

4. The above-introduced classification indicates on the following connection between the processes of birth and disappearance of defects of different levels:

$$
\begin{aligned}
&\frac{\partial T_i}{\partial x_i} = T_{ii}, \quad \frac{\partial(-2\omega_i)}{\partial x_i} = \Xi_{ii},\\
&\frac{\partial T_{ij}}{\partial x_j} = T_{ijj}, \quad \frac{\partial \Xi_{ij}}{\partial x_j} = \Omega_{ijj},\\
&\frac{\partial T_{ijk}}{\partial x_k} = T_{ijkk}, \quad \frac{\partial \Omega_{ijk}}{\partial x_k} = \Theta_{ijkk},\\
&\frac{\partial T_{ijkq}}{\partial x_q} = 0 (= T_{ijkqq}), \quad \frac{\partial \Theta_{ijkq}}{\partial x_q} = 0.
\end{aligned}
$$

One of major properties of defects in the general hierarchy is that the defects of Nth rank are the only possible sources or discharges for the defects of $(N+1)$st rank.

7. Conclusions

A new general theory of defects in continuous media is introduced in the present paper. It is shown that the defects of all known types can be described in the framework of the presently developed theory (or classification) of defects.

1. Within this unified classification the models of continuous media that allow presence of potential of displacements are interpreted as models of continuous media free of defects (the Cauchy continuous media). The defects of zero rank are the discontinuities (jumps) in the potential of displacements. The source of defects of zero level is the vector of curls (the tensor of first rank).
2. In the framework of the introduced unified classification of defects dislocations are defects of the first rank (the Papkovich–Cosserat continuous media). These media allow a two-level system of defects, the scalar defects D^1, D^2 and the dislocations D_i^2. And in this case the dislocations are the conserved defects, i.e., they cannot be born or disappear. The source of dislocations is a tensor of second rank.

3. It is shown that the second level models (the Saint-Venant continuous media) have a complex structure. They incorporate three types of defects. The source of defects in this case is a tensor of a third rank. The following structure of defects of the second level is established:
   - Disclinations can be attributed to the classical defects. The source of disclinations is the scew-symmetric part of tensor of a third rank ($T_{ijk}-T_{jik}$)—the anti-symmetric tensor in first two indexes i, j. The disclinations are defects related to discontinuities (jumps) in the field of curls.
   - Classification allows to predict the existence of two other types of defects in the continuous media of the second level. Their sources are determined through the symmetrical part of tensor of a third rank ($T_{ijk}+T_{jik}$)—the symmetric tensor in first two indexes i, j. First of them describes the discontinuities (jumps) in the change of volume, it is defined as a vector of "cavitation" T_{kkj}. Second type of these new defects of second rank describes the discontinuities (jumps) in the components of deviatoric part of strain tensor. The source of defects of a second type is a tensor of a third rank. It is defined as a tensor of "twinning", $\frac{1}{2}T_{ijk}+\frac{1}{2}T_{jik}-\frac{1}{3}T_{qqk}\delta_{ij}$.

   We called the defects of second level as the generalized disclinations.
4. The possibility of existence of defects of higher than second level is established. The necessity of existence of defects of third level is defined by the condition of generation of defects of the second level. The source of these defects is a tensor of forth rank, and they have a complex hierarchy within the their own class.
5. The classification of defects is generalized for the defects of any finite level. It is shown that the existence of defects of Nth level is necessarily determined by a possibility of generation of defects of $(N-1)$st level.
6. The introduced classification allows to describe the set of arguments of a functional in developing the mathematical continuous media models of a various complexity by means of the variational method:
   - In mathematical formulation of the Cauchy continuum media model the main kinematic variables in defining the Lagrangian of this model are the sufficient times differentiable fields D_0 and D^1_i.
   - In mathematical formulation of the Papkovich–Cosserat continuum media models the main kinematic variables in defining the Lagrangian of these models are D_0, D^1_i, D^2_{ij}. The set of the Papkovich–Cosserat media are described in the general case by 13 degrees of freedom, i.e., by the continuous fields $r^0_i = \left(\frac{\partial D^0}{\partial x_i}+D^1_i\right)$, D^2_{ij}, or otherwise ω^{Ξ}_k, θ^{Ξ}, and γ^{Ξ}_{ij}. The set of the Papkovich–Cosserat media contains in itself the sub-sets of the "classical" Cosserat media with six degrees of freedom r^0_i, ω^{Ξ}_k, as well as the media with "porosity" with only four degrees of freedom r^0_i, θ^{Ξ}, and the media with "twinning" with eight degrees of freedom r^0_i, γ^{Ξ}_{ij}, and finally the classical (Cauchy) media with three degrees of freedom r^0_i.
   - In mathematical formulation of the Saint-Venant continuous media models it is necessary to keep in mind that in the general case these models are described by 40 degrees of freedom, $D_0, D^1_i, D^2_{ij}, D^3_{ijk}$. Without account of the scalar defects, these media models allow a two-level system of defects. The first level of defects corresponds to dislocations that may be conserved, as well as they may be borne or disappear. The second level of defects corresponds to the conservable disclinations, cavitation and twinning. The set of Saint-Venant's continuous media contains in itself the sub-sets of the Papkovich–Cosserat media with 12 degrees of freedom r^0_i, ω^{Ξ}_k, θ^{Ξ}, γ^{Ξ}_{ij}, as well as the classical (Cauchy) media with three degrees of freedom r^0_i.
7. The choice of certain type of kinematic structure of the continuous medium with defects is determined by a requirement to describe certain physical properties of medium under study. For example, the models constructed on the basis of the Cauchy media principally cannot be used for developing a theory of fine dispersed composite materials. Indeed, the fine dispersed inclusions can be treated as dislocations in the parent phase or in the matrix. The same is true for the poorly degassed matrix. In this case the gas bubbles can be treated as vacancies. Such dislocations cannot be born or disappear. Therefore the theory of fine dispersed composite materials can be developed only on the basis of the model of continuum media

with defects of the first level with conserved dislocations. If the phase transitions take place in the continuous medium then they can be connected with the birth of defects—dislocations in the parent phase. It is wrong to attempt to develop such a medium model on the basis of the Papkovich–Cosserat continuous media. As a minimum, the required model in such case will be the Saint-Venant continuous media model with the generatable dislocations and conserved disclinations.

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